MathPath 2024 Qualifying Test
Instructions

Do NOT start work on these problems until you have read the instructions, and both you and a parent or guardian have also read the Certification Form below, which summarizes the rules about outside resources. Later, just before you submit your work, you will sign the Certification Form in the presence of your parent or guardian.

Encouragement
This Qualifying Test (QT) is supposed to include some parts that are very accessible, and other parts that are quite challenging. We do not expect anyone to solve all the problems correctly! In fact, we regularly admit applicants who write down interesting ideas and partial answers, but who don't get that many problems completely correct. (We also don't reveal QT scores; please don't ask.)

Electronic calculations
Except where explicitly disallowed, you may use a calculator or computer to perform calculations you find tedious, but you must say so each time. For instance, if for some reason you decide you need to compute 1.234/2.345, you can write

$$1.234/2.345 = .5262 \text{ (calculator)}.$$

If you use a computer program to get an answer by brute force (trying every possibility), you won't get much credit, even if the answer is right.

Please see the online instructions (www.mathpath.org/QTclarifications) **for:**

- **Rules for writing up your answers**
- **How to write well**, and
- **Where and how to submit**

You may want to read these instructions *before* you begin so that you don't need to reformat your work later.

MathPath 2024 Qualifying Test
Certification Form

To be read promptly, then reread and signed just before submitting your finished work.

I, _____, applicant to MathPath 2024, certify that:
print full name

1. This submission is entirely my own work. I have discussed my work on specific problems with no one except the Past Executive Director, who responds to email about the interpretation of the test questions, and anyone he gave permission for me to ask (whom I've listed below). No one but myself has reviewed or edited this submission, nor have I even shown it to anyone, prior to submission.

2. Except where explicitly permitted within a QT problem, I have not looked at any resources, including online resources, in an effort to find out background information about any of the problems on this test. Nor have I tried to get any other help online, except from the MathPath website itself. (It is possible that you will come across information relevant to this test in the course of your normal math activities. As long as you were not seeking out test help, this is OK, but it still should be reported under #3 below.)

3. If for any reason I come across information that helps me solve any of the problems, or if I had already seen a problem very similar to a QT problem, I have listed those problems below, and in my solution for each such problem I have reported what information I found or remembered. (For instance, perhaps you remembered the statement of a key theorem but not how to prove it; or perhaps you remembered the solution method but not the answer.)

4. I understand that it is plagiarism if I learn how to solve a problem from some source and then submit a solution along those lines without crediting the source. It makes no difference if I copy from that source word for word or use my own words; if the ideas come from another source, it is plagiarism if no attribution is given.

5. I understand that if MathPath staff find evidence that I have been untruthful in this Certification, that is grounds for denying admission or sending me home with no refund if MathPath 2024 is already in session.

Problem numbers of exceptions in item 3: _____

Person(s) I discussed QT with by permission: _____

To confirm this Certification, after finishing my solutions I have signed my name, and my parent or guardian has printed/signed their name, and dated this document, as my witness.

Applicant signature: _____

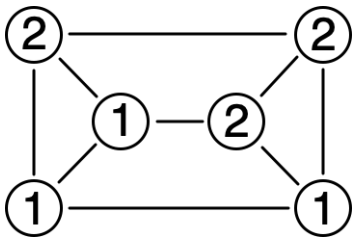
Parent/guardian printed name: _____

Signature: _____

Date: _____

MathPath 2024 Qualifying Test
Problems

1. The diagram at right is an arrow puzzle. Your task is to turn each of the nine segments between the circles into an arrow (pointing towards one circle or the other) in such a way that the total number of arrows pointing at any circle is equal to the number inside the circle. How many ways are there to solve the puzzle?



2. The Mad Hatter places a hat onto the head of each of five people sitting in a circle, while they have their eyes closed. Each hat is either blue, red, yellow, green, or purple. However, not every color has to appear—some people might have the same color hat. After opening their eyes, each person can see the colors of all the hats except their own. The Mad Hatter then states that all five hats are different colors.

- The first person looks around and states that this might be true, or it might not, but she can't tell, based on what she can see.
- The second person looks around and also states that this might be true, or it might not, but he can't tell, based on what he can see.
- The third person looks around, and states that the Mad Hatter is definitely lying.

At this point, which of the five people can deduce the color of their hat, based on what they can see and what has been said?

3. Given a positive integer M, consider the following sequence of steps.

- List all the divisors of M, including 1 and M itself, then compute their sum. For instance, when $M = 14$ the sum would be $1 + 2 + 7 + 14$.
- Next list the *reciprocals* of the divisors, and calculate their sum as well. Thus when $M = 14$ this sum would be $\frac{1}{1} + \frac{1}{2} + \frac{1}{7} + \frac{1}{14}$.
- Finally, compute the first sum above divided by the second sum.

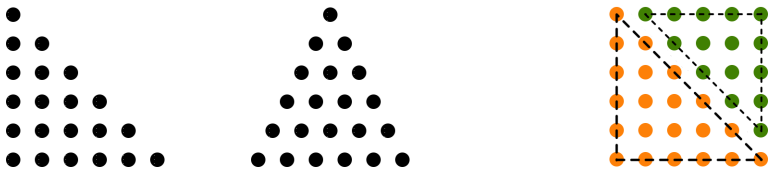
(a) Perform this process for $M = 10$. What do you notice?

(b) Determine the result when $M = pq$ is a product of two primes. (Show all steps.)

(c) Find the outcome when $M = 2^k$ is a power of 2. (Show all steps.)

(d) By now you have noticed something interesting about the result. Prove that this occurs no matter what positive integer M we start with.

4. The quantity $1+2+3+\cdots+n$ is known as a *triangular number,* and written as T_n. For instance, the sixth triangular number is $T_6 = 1+2+3+4+5+6 = 21$. We can visualize T_6 via a triangular array of dots, such as either of the diagrams on the left.



By stacking an upside down T_5 triangle on top of a T_6 triangle, as shown on the right, we see that $T_5 + T_6$ must be a square number, without even performing the calculation. (In fact the sum must be 6^2, because the T_6 triangle has six dots across its base.)

(a) Show visually that $1+5+9+13+\cdots+49$ is a triangular number. (Don't actually calculate this quantity; just draw a convincing diagram.)

(b) Show visually that n^2+2T_n is a triangular number, and figure out which triangular number it is. (Don't use algebra for this question; just draw a convincing diagram.)

(c) Show visually (without algebra) that $6T_n + 10T_{n+1}$ is a triangular number. Conjecture a generalization to this statement, and indicate why you believe that it's true.

5. Suppose that we are given 30 points in the plane, no three located on a single line, and that 10 are colored red, another 10 are colored yellow, and the other 10 are colored blue. Prove that it is possible to choose a red point A, a yellow point B, and a blue point C such that none of the remaining 27 points lie within triangle ABC.

6. Given some positive real number x, let $\lfloor x \rfloor$ denote the integer part of x (before the decimal point), and let $\langle x \rangle$ be the decimal part (after the decimal point). For instance, $\lfloor 7.89 \rfloor = 7$, while $\langle 7.89 \rangle = .89$. Next, starting with a positive real number a_0, define a sequence via

$$\boxed{a_{k+1} = \lfloor a_k \rfloor \langle a_k \rangle + 1}$$

for all $k \geq 0$. Hence if $a_0 = 7.89$, then

$$a_1 = (7)(.89) + 1 = 7.23, \qquad a_2 = (7)(.23) + 1 = 2.61, \qquad \ldots$$

Show that regardless of the initial value a_0, the sequence eventually stabilizes, meaning that $a_n = a_{n+1} = a_{n+2} = \cdots$ for some n onwards.

MathPath 2024 Qualifying Test
Essay Questions

Please answer each question below in one paragraph. Your responses will not affect your score on this Qualifying Test, but they are an important part of your application.

E1 Of your work on this qualifying test, what are you most proud of, and why?

E2 Which problem(s) on this Qualifying Test did you find most challenging, and why?

E3 While you were working on this QT, what is something that surprised you or that you found particularly interesting? Or, what is something that you're wondering about after working on this QT?

To obtain clarification on a QT problem, please contact Dr. V (Past Executive Director) at sam.vandervelde@mathpath.org. In response to questions we receive about the QT, we occasionally publish clarifications. All clarifications to date can be found at mathpath.org/QTclarifications.