# MathPath 2025 Qualifying Test

*Do NOT start work on these problems until you have read the full instructions and both you and a parent or guardian have also read the Certification Form on the next page. After you have completed the Qualifying Test, you will sign the Certification Form in the presence of your parent or guardian just before you submit.*

*Instructions viewable online at: https://www.mathpath.org/apply/applynow/instructions#qualifyingtest*

*Instructions downloadable at: https://www.mathpath.org/public/files/QTinstructions2025.pdf*

# MathPath 2025 Qualifying Test
# **Certification Form**

*To be read promptly, then reread and signed just before submitting your finished work.*

I, ________________________________, applicant to MathPath 2025, certify that:
print full name

1. This submission is entirely my own work. I have discussed my work on specific problems with no one except: the Author, who responds to email about the interpretation of the test questions; the Director of Admissions, who responds to email about general application questions; and anyone they gave permission for me to ask (whom I've listed below). No one but myself has reviewed or edited this submission, nor have I even shown it to anyone, prior to submission. I have not used AI to assist with composing this submission.

2. Except where explicitly permitted within a QT problem, I have not looked at any resources, including online resources, in an effort to find out background information about any of the problems on this test. Nor have I tried to get any other help online, except from the MathPath website itself. (It is possible that you will come across information relevant to this test in the course of your normal math activities. As long as you were not seeking out test help, this is OK, but it still should be reported under #3 below.)

3. If for any reason I come across information that helps me solve any of the problems, or if I had already seen a problem very similar to a QT problem, I have listed those problems below, and in my solution for each such problem I have reported what information I found or remembered. (For instance, perhaps you remembered the statement of a key theorem but not how to prove it; or perhaps you remembered the solution method but not the answer.)

4. I understand that it is plagiarism if I learn how to solve a problem from some source and then submit a solution along those lines without crediting the source. It makes no difference if I copy from that source word for word or use my own words; if the ideas come from another source, it is plagiarism if no attribution is given.

5. understand that if MathPath staff find evidence that I have been untruthful in this Certification, that is grounds for denying admission or immediate dismissal from the program with no refund if MathPath 2025 is already in session.

Problem numbers of exceptions in item 3: ____________________________

Person(s) I discussed QT with by permission: ____________________________________

To confirm this Certification, after finishing my solutions, I have signed my name, and my parent or guardian has printed/signed their name, and dated this document, as my witness.

Applicant signature: ________________________________

Parent/guardian printed name: ________________________________

Signature: ________________________________

Date: ________________________________

# MathPath 2025 Qualifying Test
# **Problems**

1. Starting with a positive fraction $\frac{a}{b}$ in lowest terms*, we "shred" this fraction by taking the following steps:

1. Multiply it by $\frac{2}{3}$
2. Reduce if possible
3. Add 1 to the numerator and add 1 to the denominator
4. Reduce if possible.

For instance, if we shred $\frac{6}{19}$ we get

$$\frac{6}{19} \xrightarrow{\text{times } \frac{2}{3}} \frac{12}{57} \xrightarrow{\text{reduce}} \frac{4}{19} \xrightarrow{\text{add 1s}} \frac{5}{20} \xrightarrow{\text{reduce}} \frac{1}{4}.$$

We say a fraction is "shreddable" if performing the steps above produces a different fraction than we started with.

(a) Find two different fractions (other than $\frac{6}{19}$) that result in $\frac{1}{4}$ when shredded.

(b) Find a shreddable fraction $\frac{m}{n}$ that produces a new fraction when shredded, but when we shred this new fraction, we get back to $\frac{m}{n}$. Demonstrate that your choice actually works.

2. For each positive integer $n$, let's separate the numbers from 1 to $n$ into sets so that:

- the sum of the numbers in each set is a power of 2 (that is, one of 1, 2, 4, 8, ...)†,
- each number is in a set, and
- no number is in more than one set.

For instance, we can do so for $n = 6$ by making the sets $\{1\}$, $\{2, 6\}$, $\{4\}$, and $\{3, 5\}$.

(a) Demonstrate how this can be done for $n = 11$.

(b) Prove that this is possible for every positive integer $n$.

(c) Is it always possible to split the numbers from 1 to $n$ into sets whose sums are *distinct*‡ powers of 2? Either explain why or give a counterexample.

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∗ "Lowest terms" means that the fraction cannot be reduced.

† A power of 2 is a number that can be written in the form $2^n$ (that is, $1 = 2^0, 2 = 2^1, 4 = 2^2, \ldots$).

‡ "Distinct" means that they are all different.

3. Let's draw seven lines in the plane with the following rules:

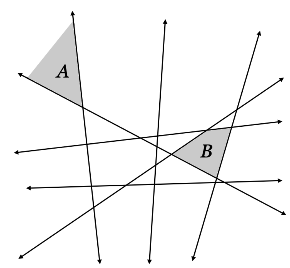
- no two of the lines are parallel, and
- no three lines pass through (or intersect at) a single point.

An example of seven such lines is shown, but the questions below are about any such set of seven lines. (Note that not all intersection points and regions appear in this diagram.)

(a) Some of the regions (such as region $A$) extend out infinitely far, while others (like region $B$) are bounded. How many of the regions are infinite? Briefly explain your answer.

(b) Prove that, no matter how we draw the seven lines (according to the rules), there is an angle formed by two of the lines that is less than 26°.

(c) Show that, no matter how we draw the seven lines (according to the rules), at least one of the bounded regions must be a triangle.



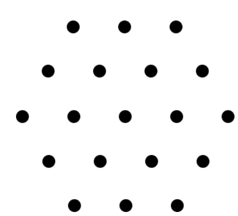

4. A *magic rectangle* is a $2 \times 3$ grid filled with positive integers with the following rules:

- the sums of the numbers in each row are equal to each other, and
- the sums of the numbers in each column are also equal to each other.

Note that the sums of the rows do not have to match the sums of the columns. A magic rectangle is shown here.

| | | |
|---|---|---|
| 1 | 4 | 4 |
| 5 | 2 | 2 |

A *hexagonal dot array* of size $k$ is a triangular lattice of dots in the shape of a hexagon whose outer sides are $k$ dots long. A size 3 hexagonal dot array appears at right.



(a) By making an organized count of them all, show that the number of magic rectangles with column sum 6 is the same as the number of dots in a size 3 hexagonal dot array.

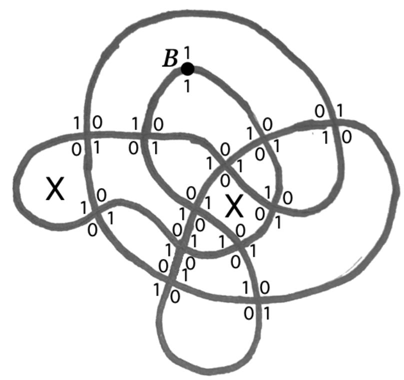
(b) Prove that the number of magic rectangles with column sum $2k$ is always equal to the number of dots in a size $k$ hexagonal dot array.

5. Create a figure using the following steps:

1. From a base point $B$, draw a loop that begins and ends at $B$. The loop may cross (or intersect) itself (but not at $B$) to create regions, and it should always cross itself at approximately right angles.
   (Notice that the loop cannot have a multi-crossing because that would result in angles much smaller than right angles. The loop cannot be infinite so there will be a finite number of regions created.)
2. Next, write the number 1 once on either side of point $B$, as shown in the example.
3. Finally, at each crossing, write the numbers 0, 1, 0, 1 (in that order) within the corners of the regions that meet there. (You can pick any corner to start, but then you must alternate 0, 1, 0, 1 as you go around the crossing so that the 0s are not next to each other and the 1s are not next to each other.)

(a) In the example shown, the sum of the 0s and 1s within each region bounded by the loop is even, except for the two regions marked by an X. Find a different way (following the rules!) of writing 0s and 1s so that only one enclosed region has an odd sum. (The region completely outside of the loop does not count as an enclosed region.)

(b) Prove that no matter how the loop is drawn, it is impossible for every region to have an even sum.



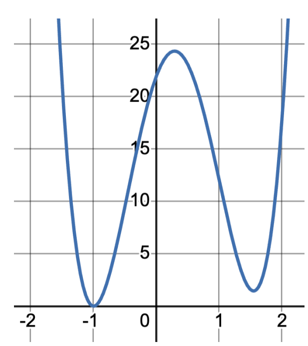

6. The graph (on a traditional coordinate plane) of $y = 9x^4 - 10x^3 - 25x^2 + 16x + 22$ is provided. We want to prove that the graph never goes below the $x$-axis, using the following strategy. Our strategy is to write $9x^4 - 10x^3 - 25x^2 + 16x + 22$ in the form

$$A\,(p(x))^2 + B\,(q(x))^2,$$

where $A$ and $B$ are positive real numbers§, while $p(x)$ and $q(x)$ are quadratic functions¶.

(a) Show that $(x+1)$ must be a factor of both $p(x)$ and $q(x)$.

(b) Find an expression $A\,(p(x))^2 + B\,(q(x))^2$ that is equal to $9x^4 - 10x^3 - 25x^2 + 16x + 22$. How does this complete the proof that the graph never goes below the $x$-axis?




§ A real number is any value, rational or irrational, on the number line.

¶ A quadratic function is in the form $ax^2 + bx + c$

# MathPath 2025 Qualifying Test
# **Essay Questions**

*Please answer each question below in one paragraph. Your responses will not affect your score on this Qualifying Test, but they are an important part of your application.*

**E1** Of your work on this qualifying test, what are you most proud of, and why?

**E2** Which problem(s) on this Qualifying Test did you find most challenging, and why?

**E3** While you were working on this QT, what is something that surprised you or that you found particularly interesting? Or, what is something that you're wondering about after working on this QT?

To obtain clarification on a QT problem, please contact Dr. V (the Author) at sam.vandervelde@mathpath.org. In response to questions we receive about the QT, we occasionally publish clarifications. All clarifications to date can be found at mathpath.org/QTclarifications.